FUZZY DE MORGAN ALGEBRA

"It is not certain that everything is uncertain."
Blaise Pascal, French mathematician

10/27/2016

INTRODUCTION

A "Fuzzy set" is defined as a class of objects with a continuum of grades of membership. It is characterized by a "membership function" or "characteristic function" that assigns to each member of the fuzzy set a degree of membership in the unit interval [0, 1].

Some examples of fuzzy sets are:

1. The class of all real numbers which are **much greater than** unity.
2. The class of **tall** people.
3. The class of **operational** pumps.
4. The class of **adequate** flow rates.
5. The class of **unacceptable** fuel temperatures.

The notion of a fuzzy set is non-statistical in nature.

DEFINITION

Let X be a space of points or objects:

$$X = \{x\},$$

where x is a generic element of X.

A fuzzy set (class) A in X is characterized by a membership (characteristic) function:

$$\mu_A(x),$$

which associates with each point in X a real number in the unit interval [0, 1], with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A.

EXAMPLE

Figure 1 shows the membership function that describes the fuzzy variable:

$$A \equiv \{\text{class of } \textit{unacceptable} \text{ nuclear fuel temperatures}\}$$

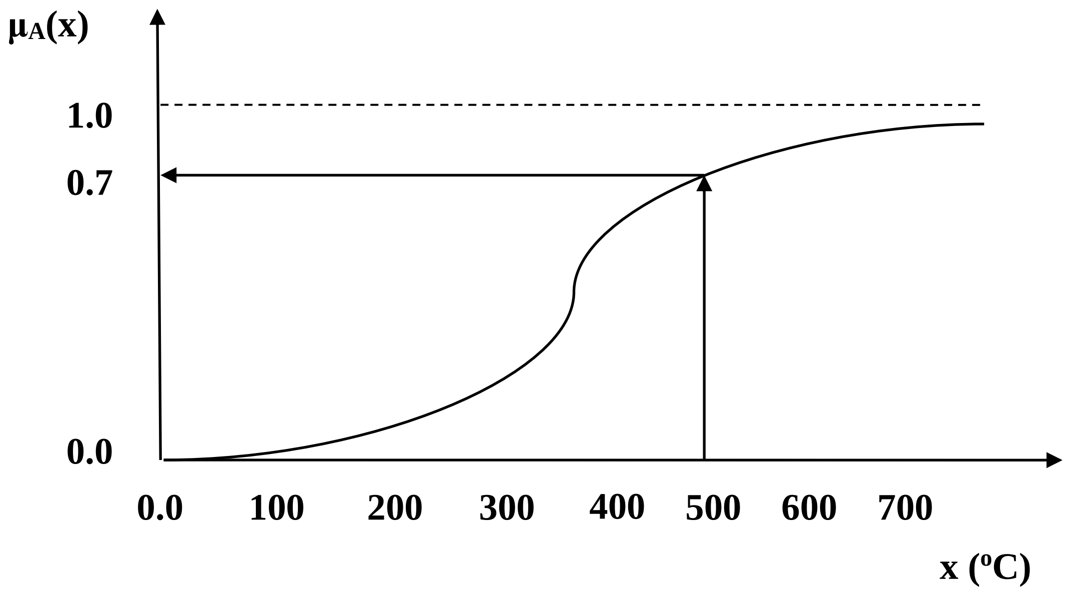

Figure 1. Membership function describing the class of unacceptable nuclear fuel temperatures. The x = 500 ºC temperature belongs to the class of unacceptable fuel temperatures to a degree of 0.7.

In ordinary set theory the degree of membership is Boolean in nature:

$$\mu_A(X) = 1, \ \forall \text{x} \in \text{A (belongs to A)}$$
$$= 0, \ \forall \text{x} \notin \text{A (does not belong to A)}$$

DEFINITIONS

A fuzzy set is "empty" when its membership function is identically zero over X.

$$\mu_A(X) = 0, \ \forall x \in X \qquad (1)$$

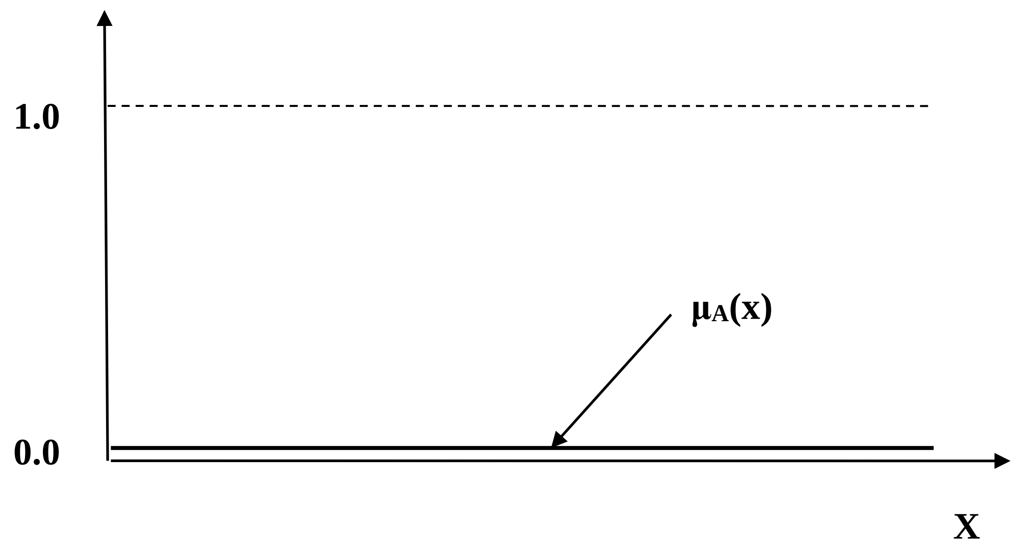

Figure 2. Graphical representation of a fuzzy empty set.

Two fuzzy sets A = B are "equal" iff (if and only if; a necessary and sufficient condition):

$$\mu_A(X) = \mu_B(X),\ \forall x \in X \qquad (2)$$

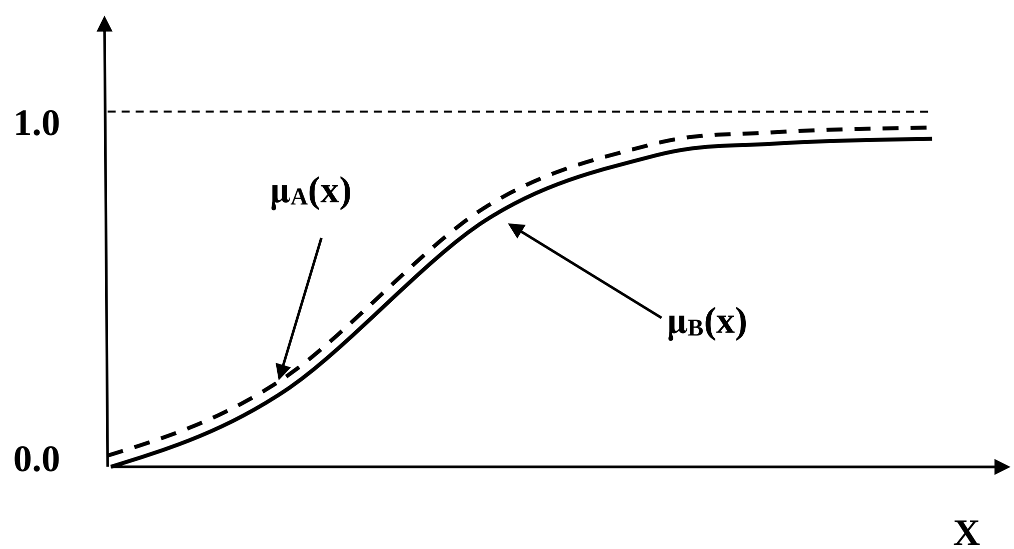

Figure 3. Graphical representation of two equal fuzzy sets.

The “complement” of a fuzzy set A is defined as:

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \quad (3)$$

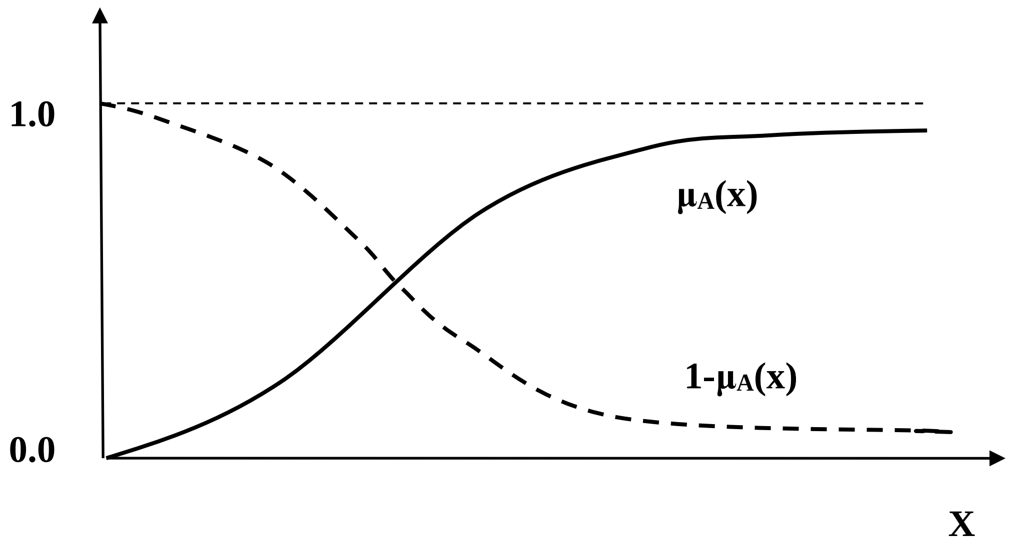

Figure 4. Graphical representation of the complement of a fuzzy set.

The “containment” of a fuzzy set A in another fuzzy set B is defined as:

$$A \subset B \Leftrightarrow \mu_A \leq \mu_B \quad (4)$$

This can also be expressed as:

“A is contained in B”
“A is a subset of B”
“A is less or equal to B”

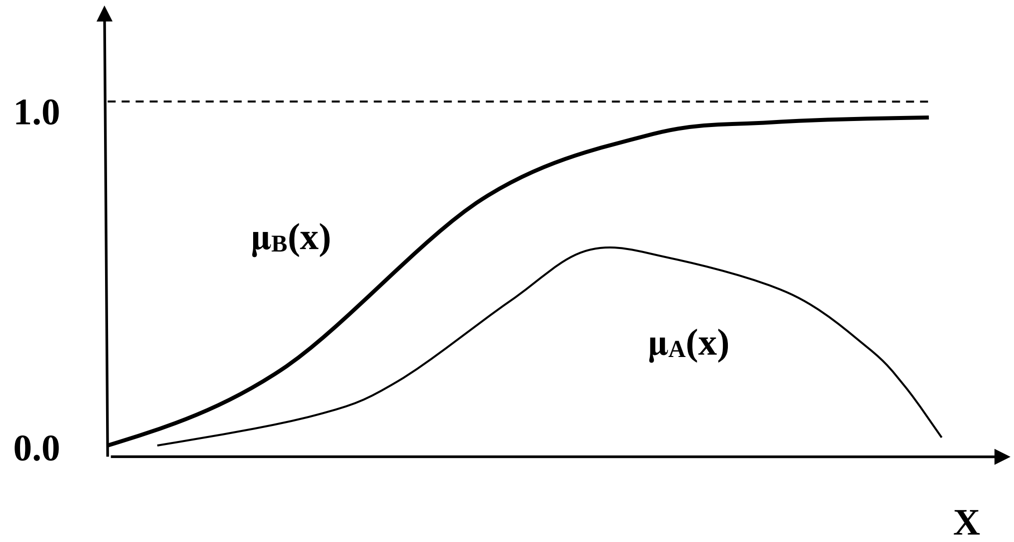

Figure 5. Graphical representation of the containment of a fuzzy set A in another fuzzy set B.

The “union” of two fuzzy sets A OR B is described as:

$$
\begin{aligned}
&C = A \cup B = A\,OR\,B \\
&\mu_C = \mu_A \vee \mu_B \\
&\mu_C(x) = Max[\mu_A(x), \mu_B(x)], \forall x \in X
\end{aligned}
\tag{5}
$$

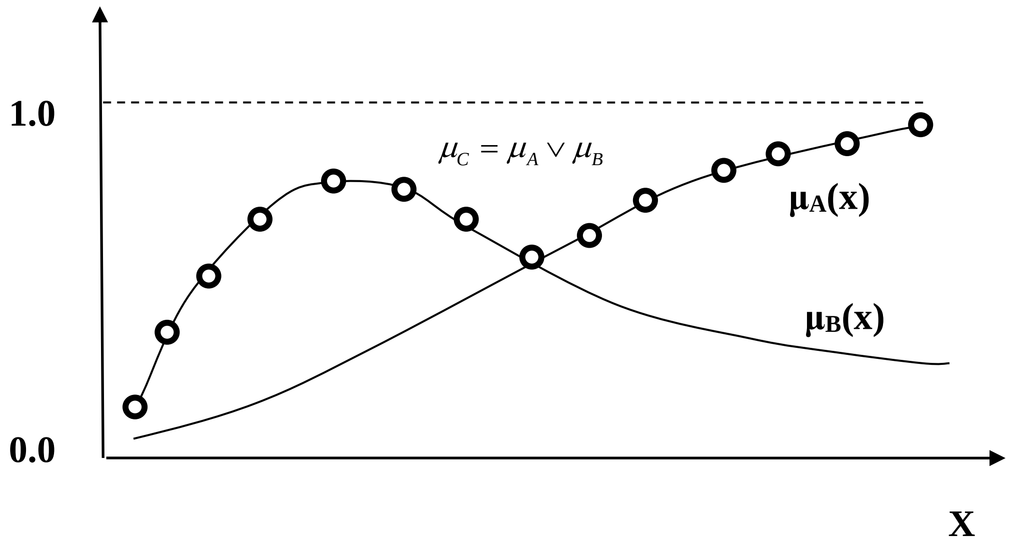

Figure 6. Graphical representation of the union of two fuzzy sets A and B.

The "intersection" of two fuzzy sets A AND B is described as:

$$
\begin{aligned}
&C = A \cap B = A\, AND\, B \\
&\mu_C = \mu_A \wedge \mu_B \\
&\mu_C(x) = Min[\mu_A(x), \mu_B(x)], \forall x \in X
\end{aligned}
\qquad (6)
$$

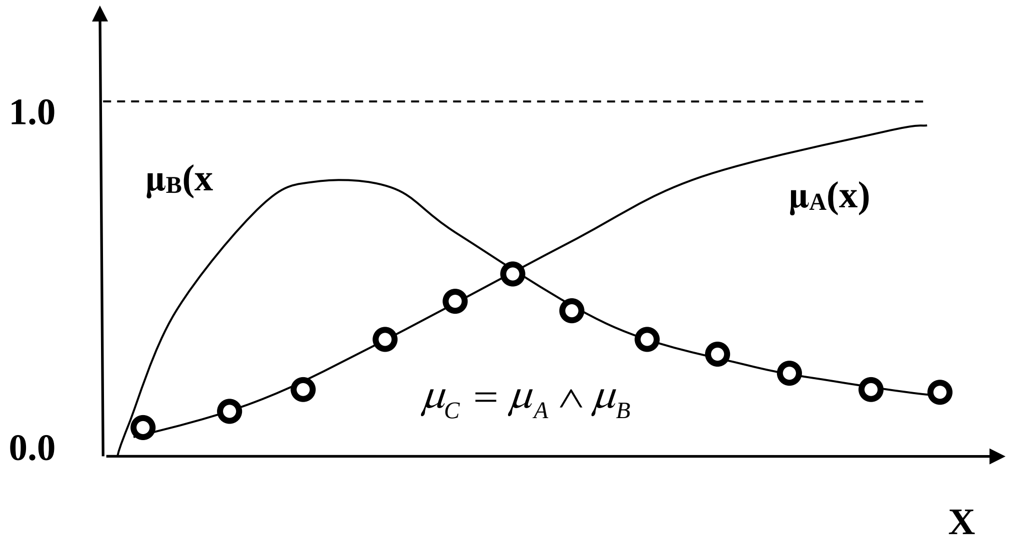

Figure 7. Graphical representation of the intersection of two fuzzy sets A and B.

Two fuzzy sets A, B are "disjoint" if their intersection is "empty", or equal to the null set:

$$A \cap B = \phi \tag{7}$$

FUZZY, DE MORGAN ALGEBRA

A "Fuzzy Algebra" or a "de Morgan Algebra", is a set z that has at least two distinct elements:

$$Z = (z, +, *, \rightarrow) \tag{8}$$

with the two binary operations:

+ conjunction

* disjunction

And a unary operation:

$\rightarrow$ implication

Whenever z has more than two elements, there is not a unique complement such that:

$$x * \bar{x} = 0, \quad x + \bar{x} = 1 \tag{9}$$

the definition of the identities.

FUZZY BINARY OPERATIONS

The following binary operations are defined over a Fuzzy Algebra:

$$\begin{array}{lll} \text{Union} & \text{C=A} \cup \text{B, C=A OR B,} & \mu_C = \mu_A \vee \mu_B \\ \text{Intersection} & \text{C=A} \cap \text{B, C=A AND B,} & \mu_C = \mu_A \wedge \mu_B \\ \text{Complementation} & \bar{\text{A}}, & \mu_{\bar{A}} = 1 - \mu_A \end{array} \tag{10}$$

AXIOMS OF FUZZY LOGIC AND FUZZY ALGEBRA

The axioms, laws or postulates of a Fuzzy Algebra are similar to those of a Boolean Algebra.

The following axioms or laws apply:

$$\begin{array}{ll} \text{Idempotency law} & x \cup x = x \\ & x \cap x = x \\ \text{Commutativity law} & x \cup y = y \cup x \\ & x \cap y = y \cap x \\ \text{Associativity law} & (x \cup y) \cup z = x \cup (y \cup z) \\ & (x \cap y) \cap z = x \cap (y \cap z) \\ \text{Absorption law} & x \cup (x \cap y) = x \\ & x \cap (x \cup y) = x \\ \text{Distributivity law} & x \cup (y \cap z) = (x \cup y) \cap (x \cup z) \\ & x \cap (y \cup z) = (x \cap y) \cup (x \cap z) \\ \text{De Morgan law} & (\overline{x \cup y}) = \bar{x} \cap \bar{y} \\ & (\overline{x \cap y}) = \bar{x} \cup \bar{y} \end{array} \tag{11}$$

COMPARISON TO ORDINARY SET THEORY

In a Boolean Algebra, the law of the "excluded middle" which defines the universal bounds or identities, applies:

$$\begin{array}{l} A \cap \bar{A} = A \, AND \, \bar{A} = 0 \\ A \cup \bar{A} = A \, OR \, \bar{A} = 1 \end{array} \tag{12}$$

This law does not apply in the case of a Fuzzy Algebra, allowing for the inclusion of the middle point:

$$A \cap \bar{A} = A\,AND\,\bar{A} \neq 0$$
$$A \cup \bar{A} = A\,OR\,\bar{A} \neq 1 \qquad (13)$$

SETS AS POINTS

In an interpretation of fuzzy sets by Kosko, they are considered as points in the unit interval, unit square, unit cube and in general in a hypercube or n-dimensional cube.

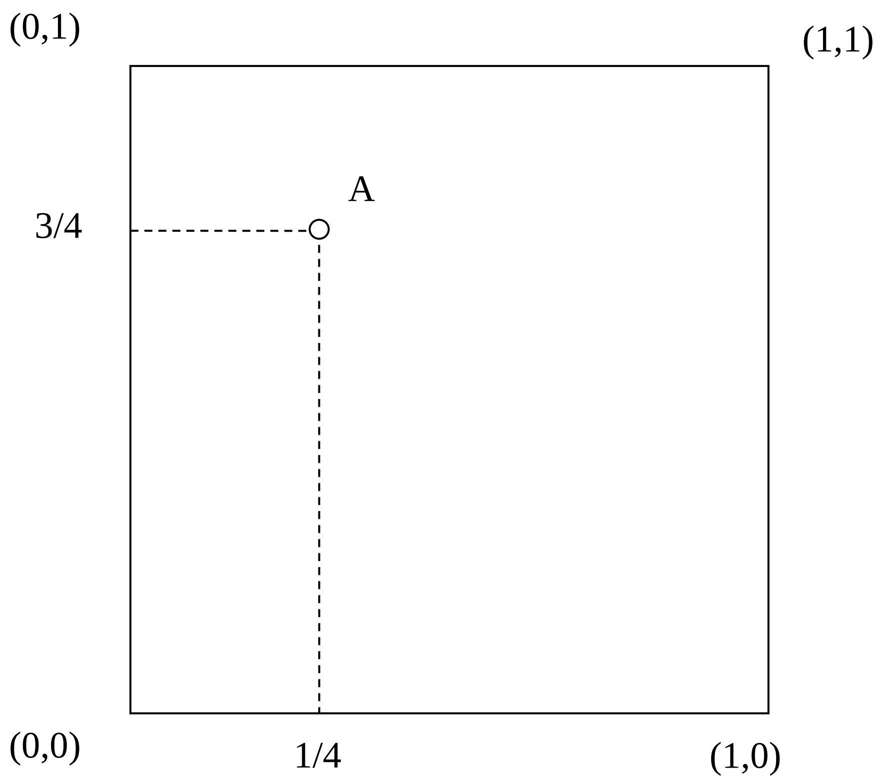

Figure 8. The set {1/4, 3/4} represented as the point A in the unit cube.

If we consider the point A in the unit cube, its complement can also be represented as a point, as well as the union and the intersection of A and its complement:

$$A = \{\frac{1}{4}, \frac{3}{4}\}$$

$$\bar{A} = \{1 - \frac{1}{4}, 1 - \frac{3}{4}\} = \{\frac{3}{4}, \frac{1}{4}\}$$

$$A \cup \bar{A} = \{Max(\frac{1}{4}, \frac{3}{4}), Max(\frac{3}{4}, \frac{1}{4})\} = \{\frac{3}{4}, \frac{3}{4}\}$$

$$A \cap \bar{A} = \{Min(\frac{1}{4}, \frac{3}{4}), Min(\frac{3}{4}, \frac{1}{4})\} = \{\frac{1}{4}, \frac{1}{4}\}$$

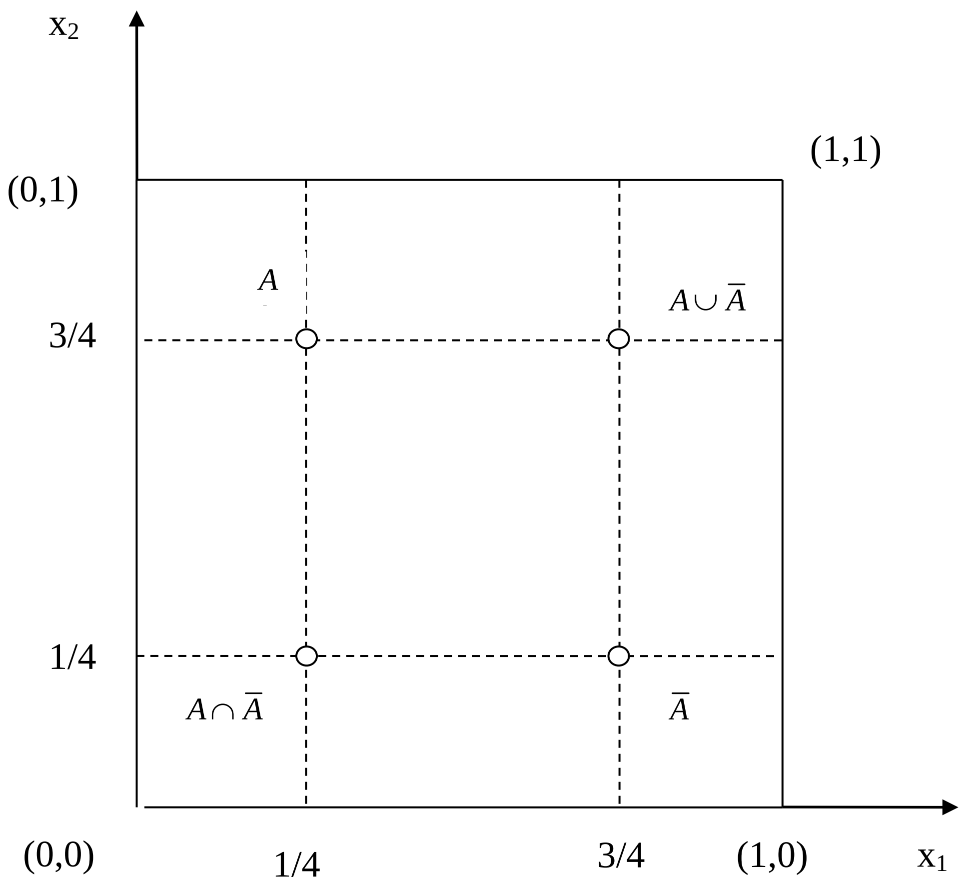

Figure 9. The point A representing the set {1/4, 3/4} and its complement in the unit square, as well as their union and intersection.

It must be noticed that fuzzy logic considers all the points in the unit square, whereas ordinary set theory only considers the corners of the square.

EXERCISE

1. Use Kosko's interpretation of fuzzy sets as points in the hypercube to graphically show:
a) In the unit square, the fuzzy set, A:{1/4,3/4}, A’, A OR A’, A AND A’.
b) In the unit cube, the fuzzy set, A:{1/4,1/2,3/4}, A’, A OR A’, A AND A’.
c) For the case of the four dimensional hypercube set, A:{1/4, 1/3, 1/2, 3/4} calculate A’, A OR A’, A AND A’.