Vector Relations

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Element of Volume

Cartesian coordinates

$$dV_{car} = dx\,dy\,dz$$

Cylindrical coordinates

$$dV_{cyl} = rdr\,d\theta\,dz$$

Spherical coordinates

$$dV_{sph} = r^2 \sin\theta\,dr\,d\theta\,d\varphi$$

Gradient

The gradient operates only on a scalar function ϕ and converts it into a vector. The gradient of a scalar function is the vector whose components are the rates of change of the function along the direction of the component.

Cartesian coordinates

$$grad_{car}\,\phi = \nabla_{car}\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}$$

Cylindrical coordinates

$$grad_{cyl}\,\phi = \nabla_{cyl}\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{\partial\phi}{\partial z}\hat{z}$$

Spherical coordinates

$$grad_{sph}\,\phi = \nabla_{sph}\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}$$

Divergence

Divergence or Gauss' Theorem

The divergence operates only on a vector $\overline{F}$ and converts it into a scalar function. According to the divergence theorem, the integral of the normal component of a vector over a closed surface is equal to the integral of the divergence of the vector throughout the enclosed volume:

$$\iint_S \overline{F}.\overline{n}\,dS = \iiint_V div\,\overline{F}\,dV = \iiint_V \nabla.\overline{F}\,dV\,,$$

where: $\overline{n}$ is a unit vector normal to the surface S bounding the volume V.

Cartesian coordinates

$$div_{car}\,\overline{F} = \nabla_{car}.\overline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Cylindrical coordinates

$$div_{cyl}\,\overline{F} = \nabla_{cyl}.\overline{F} = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_r\right) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Spherical coordinates

$$div_{sph}\,\overline{F} = \nabla_{sph}.\overline{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}$$

Laplacian

This operator is the divergence of the gradient of a scalar function ϕ.

Cartesian coordinates

$$\nabla^2_{car}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Cylindrical coordinates

$$\nabla^2_{cyl}\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

Spherical coordinates

$$\nabla^2_{sph}\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$